ON SELF-SIMILAR SINGULAR SOLUTIONS OF THE COMPLEX GINZBURG-LANDAU EQUATION.

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Abstract. We address the open problem of existence of singularities for the complex Ginzburg-Landau equation. Using a combination of rigourous results and numerical computations, we describe a countable family of self-similar singularities. Our analysis includes the super-critical non-linear Schrödinger equation as a special case, and most of the described singularities are new even in that situation. We also consider the problem of stability of these singularities.

1. Introduction. In this paper we study singular solutions to the initial value problem for the complex Ginzburg-Landau equation (CGL)

$$i\frac{\partial u}{\partial t} + (1-i\epsilon)\Delta u + (1+i\delta)|u|^{2\sigma}u = f\,, \quad \text{in } \mathbb{R}^d \times (0,T)\,, \tag{1.1}$$

$$u(x,0) = u_0(x)\,. \tag{1.2}$$

where $u = u(x,t)$ is a complex-valued function defined in $\mathbb{R}^d \times (0,T)$ and satisfying suitable decay conditions as $|x| \to \infty$, the parameters ϵ, δ, σ are non-negative real numbers, u_0 is a given initial condition, which is assumed to be smooth, with suitable decay as $|x| \to \infty$, and $f = f(x,t)$ is a given function, also assumed to be smooth, with suitable decay as $|x| \to \infty$.

We are mostly interested in the case $\epsilon > 0$, $\delta \geq 0$, and $2/d < \sigma < 2/d + 1/2$. However, some of our results are new even for the case $\epsilon \geq 0$ $\delta \geq 0$, $\sigma > 2/d$, i.e., they also include the super-critical non-linear Schrödinger equation. Under the assumption $\epsilon > 0$, $0 < \sigma < 2/d + 1/2$ it is possible to prove that for each smooth u_0 and f, with an appropriate decay at infinity, the problem (1.1)-(1.2) has a *suitable weak solution*, see [2] and [17]. Such a solution is regular away from a closed set $\mathcal{S} \subset \mathbb{R}^d \times (0,T)$ with $\mathcal{P}^{d-2/\sigma}(\mathcal{S}) = 0$, where $\mathcal{P}^\alpha$ denotes the *parabolic α-dimensional Hausdorff measure*, see [17]. It has been an open problem whether the singular set $\mathcal{S}$ can be non-empty.

We present a very strong evidence, which is based on a combination of rigorous analysis and numerical computations, that singularities may indeed exist. We shall see that there appears to exist a countable set of different types of singular solutions. Among these solutions we identify (numerically) those which are stable under a suitable notion of stability defined in Section 4. It turns out that while most of the solutions are unstable, in certain cases there may exist more than one type of stable singularities.

The case of the non-linear Schrödinger equation (NLS) (i.e., $\epsilon = \delta = 0$) and $f = 0$ has been studied by numerous authors, see for example [10, 3] or the monograph [14]. In particular, Zacharov ([18]) conjectured the existence of self-similar singularities of

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the form

$$u(x,t) = (2\kappa(T-t))^{-\frac{1}{2}(\frac{1}{\sigma}+i\frac{\omega}{\kappa})}\, Q\left((2\kappa(T-t))^{-1/2}|x|\right)\,, \tag{1.3}$$

where $Q(\xi)$ is a complex valued function defined on $(0,\infty)$, with asymptotic behavior

$$Q(\xi) \sim \xi^{-1/\sigma - i\omega/\kappa}\,, \quad \text{as} \quad \xi \to \infty\,.$$

While no rigorous proof of this conjecture seems to be available, there is an overwhelming evidence based on numerical and formal analytical calculations supporting the existence of such singularities (see, e.g., [6, 7, 14]). We also refer the reader to [3] for a rigorous result supporting the conjecture.

In this paper we will argue that these singularities persist also for a certain range of $\epsilon > 0$ and $\delta > 0$. In fact, we shall find many new self-similar singularities even for the case $\epsilon = \delta = 0$ (and $f = 0$). Using the self-similar singular solutions of the form (1.3) one can easily construct singular solutions of (1.1) and (1.2) with compactly supported, smooth u_0 and compactly supported, smooth f.

From the form (1.3) of self-similar solutions one obtains the following boundary value problem for the function Q:

$$(1-i\epsilon)(Q'' + \frac{d-1}{\xi}Q') + i\kappa\xi Q' + i\frac{\kappa}{\sigma}Q - \omega Q + (1+i\delta)|Q|^{2\sigma}Q = 0\,, \tag{1.4}$$

$$Q'(0) = 0\,, \quad \text{and} \tag{1.5}$$

$$Q(\xi) \sim \xi^{-1/\sigma - i\omega/\kappa} \quad \text{as} \quad \xi \to \infty\,. \tag{1.6}$$

In general, this problem does not have a solution for arbitrary values of parameters κ, ω and hence the unknowns in (1.4)-(1.6) are Q, κ and ω.

When $\epsilon = 0$, see, for example, [14] for a discussion of (1.6). For $\epsilon > 0$ the condition $Q(\xi) \sim \xi^{-1/\sigma - i\omega/\kappa}$ at infinity is dictated by the partial regularity result mentioned earlier, since u defined by (1.3) must be regular at almost all points of the form (x,T), $x \neq 0$. The presented results are based on a detailed analysis of the boundary value problem (1.4)-(1.6). The analysis employs both analytical and numerical techniques and is naturally divided into the following two steps:

(i) First, we rigorously prove that for $\xi_1 \geq 1$ and for each set of parameter values $\epsilon \geq 0$, $\delta \geq 0$ and $2/d < \sigma < 2/d + 1/2$ there exists a two-dimensional manifold of solutions to (1.4) on (ξ_1, ∞) with the correct asymptotic behavior at infinity.

(ii) Second, we solve numerically the boundary value problem on $(0, \xi_1)$ with an appropriate (approximate) boundary condition for Q at the boundary point $\xi = \xi_1$. In this step the choice of ξ_1 is based on numerical evidence of convergence, and not on the rigorous estimates obtained in (i), which contain constants we did not try to evaluate exactly.

We briefly summarize main results of our calculations. We note a (scaling) symmetry in the problem (1.4)-(1.6): If $\lambda > 0$, $\theta \in \mathbb{R}$ and $(Q(\xi), \kappa, \omega)$ is a solution of (1.4)-(1.6) then $(\lambda^{1/\sigma + i\theta}Q(\lambda\xi), \lambda^2\kappa, \lambda^2\omega)$ is also a solution. We chose a representative in each of these families of solutions by imposing suitable normalization conditions. We mostly work with the normalization $\omega = 1$ and $\operatorname{Im} Q(0) = 0$, $Q(0) > 0$. Such solutions will be called *normalized* solutions and they are uniquely determined by two parameters (κ, μ), where $\mu = Q(0)$. Some numerical results are better presented in a

different parameterization in which $Q(0)=1$ is fixed, (κ,ω) are used as parameters. We shall state the use of the latter normalization explicitly whenever it is used.

We observe that for fixed values of $d\geq 1$, $\sigma>2/d$ and $\epsilon=\delta=0$ there exists a countable family of normalized solutions $(Q_j(\xi),\kappa_j)=(Q_j(\xi),\kappa_j,1)$, $j=1,2,\dots$ of (1.4)-(1.6). The $j-$th profile $|Q_j|$, when extended to $(-\infty,\infty)$ as an even function, has exactly j local maxima, and somewhat resembles the profile of the $j-$th state of an elementary quantum mechanical oscillator. A precise quantum mechanical interpretation of the solutions is more complicated, and is related to the so-called *resonances* or *quasi-stationary states*. The first solution of the family has been known, see [10] or [14], for example. We are not aware of any mentioning of the other solutions in the literature. All the solutions persist if ϵ and δ are perturbed to (small) strictly positive values. We now describe the behavior of these perturbed solutions for $\delta=0$. We let $\mu_j=Q_j(0)$. As mentioned above, the solution (Q_j,κ_j) is determined by (κ_j,μ_j). We observe that a branch of solutions parameterized by $(\kappa_j(\epsilon),\mu_j(\epsilon))$ emanates from each point (κ_j,μ_j). We plot the curves $(\epsilon,\kappa_j(\epsilon))$, $j=1,\dots,5$, in Figure 3.1 for $d=1$, $\sigma=2.3$ and in Figure 3.6 for $d=3$, $\sigma=1$. In the other figures we plot the profiles $|Q_j^\epsilon(\xi)|$ of the corresponding solutions $Q_j^\epsilon(\xi)$ at certain points along each branch.

An interesting feature observed in the behavior of the branches, is the existence of a turning point on each branch at $\epsilon=\epsilon_j^*$. We see from the graphs that as ϵ returns to zero along the branch, $\kappa_j(\epsilon)$ tends to zero, suggesting that the solution Q_j^ϵ converges to a (radial) solution of the equation

$$\Delta Q-Q+|Q|^{2\sigma}Q=0\,,\ \text{ in }\mathbb{R}^d\,, \tag{1.7}$$

satisfying $Q(x)\to 0$ as $|x|\to\infty$.

Our computations presented in Section 3 clarify the structure of the diagram which turns out to be slightly more complicated than the picture suggested above. We conjecture the following: If $d>1$, the solutions Q_j^ϵ corresponding to branch $(\kappa_j(\epsilon),\mu_j(\epsilon))$ with an odd index $j=2k-1$ converge (as ϵ and $\kappa_j(\epsilon)$ approach zero) to the k-th (normalized) radial solution of (1.7). We recall that the first of these solutions is usually called *the ground state*, and that for $d=1$ there are no other solutions of (1.7) satisfying the appropriate boundary conditions. (See, for example, [14] for more details.) If $d=1$ and also for $j=2k$ in the case $d>1$, as ϵ and $\kappa_j(\epsilon)$ approach zero, the profiles Q_j^ϵ separate into j approximate ground states which move away from each other. In particular, for $j=2k$ the profiles converge locally uniformly to zero. When $d=1$ and $j=2k-1$, the profiles converge locally uniformly to the ground state.

We tabulate results of our numerical calculations in Table 3.1 and Table 3.2.

We also looked at branches of solutions when δ is related to ϵ by $\delta=r\epsilon$, with $r>0$ of order 10^{-1} and 10^0. The behavior was similar to the case $\delta=0$, with the turning point ϵ^* getting closer to zero as r increased, as one might heuristically expect.

Questions related to stability of the singularities are addressed in Section 4. Our calculations indicate that for the non-linear Schrödinger equation all the new singularities we found are unstable, and the singularity corresponding to (κ_1,μ_1) is stable. The situation is more complicated for $\epsilon>0$, see Section 4 for details.

Our interest in singular solutions to CGL stems from analogies between (1.1) in the case $d=3$, $\sigma=1$, $\epsilon,\delta>0$ and the three-dimensional Navier-Stokes equation (NSE). The two equations have the same scaling properties and the same energy identity. Moreover, the existence and partial regularity theory of weak solutions for

NSE and CGL are similar (with CGL being technically easier), see [17]. The analogy between NSE and CGL may be rather superficial and may break down at any deeper level. However, at the same time there are no known properties of solutions to the Navier-Stokes equation which would prevent the same scenario as presented here for CGL.

The formula for the Navier-Stokes equation corresponding to (1.3) would be

$$u(x,t) \sim (2\kappa(T-t))^{-1/2}\, U\left((2\kappa(T-t))^{-1/2}x, \tau\right), \tag{1.8}$$

where $\tau = \frac{1}{2\kappa}\ln\frac{T}{T-t}$, and U is a suitable divergence-free vector field periodic in τ with suitable decay in the self-similar variable $y = (2\kappa(T-t))^{-1/2}x$.

The case, $\partial U/\partial\tau \equiv 0$, was already considered by Leray [8]. It was proved in [11] and in greater generality in [15] that NSE does not admit non-trivial solutions of the form (1.8) with U independent of τ. The problem is open for U periodic in τ.

We finish the introduction with the following speculation. Most of the singularities we have found are unstable, hence it is unlikely they would be observed in direct numerical simulations of the initial value problem (1.1)-(1.2) or in physical experiments that are modeled by CGL. Could it perhaps be the case that NSE does admit singular solutions (say of the form (1.8)), but *all* of them are unstable and therefore more or less impossible to be detected in direct numerical simulations or physical experiments ?

This intriguing scenario was once suggested to one of the authors by Sergiu Klainerman during a lunch-break conversation at a conference in Southern California.

2. Analysis of the profile equation at infinity. In this section we study solutions of (1.4) in the interval (ξ_1, ∞) satisfying the condition $Q(\xi) \sim \xi^{-1/\sigma - i\omega/\kappa}$ as $\xi \to \infty$. Heuristically one expects that the behavior of such solutions is mainly governed by the linear part of (1.4):

$$(1-i\epsilon)u'' + (1-i\epsilon)\frac{d-1}{\xi}u' + i\kappa\xi u' + \frac{i\kappa}{\sigma}u - \omega u = 0 \tag{2.1}$$

Equation (2.1) is equivalent to Kummer's equation, also known as the confluent hypergeometric equation. The solutions of this equation are well-understood, see for example [13], and one can hence get a more or less complete picture of the behavior of solutions to (2.1). We shall use analytical tools from the theory of confluent hyper-geometric equations to describe solutions of the full equation (1.4).

A canonical form of Kummer's equation is

$$z\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + (b-z)\frac{\mathrm{d}w}{\mathrm{d}z} - aw = 0\,, \tag{2.2}$$

and the equation (2.1) is transformed into this form by letting

$$z = \frac{-i\kappa}{(1-i\epsilon)}\frac{\xi^2}{2}\,, \qquad a = \frac{1}{2}\left(\frac{1}{\sigma} + \frac{i\omega}{\kappa}\right), \qquad b = \frac{d}{2}\,. \tag{2.3}$$

There is voluminous literature on this equation and properties of special functions (confluent hyper-geometric functions) which appear as its solutions. We recall some of the properties and, for the convenience of the reader, we also sketch how to derive them. For more details about confluent hyper-geometric functions we refer the reader to [13], [9], [16].

A classical formula for a solution of (2.2) is given by

$$U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tz} t^{a-1} (1+t)^{b-a-1}\, dt\,, \tag{2.4}$$

where Γ is Euler's gamma function. The integral is clearly well-defined for $\mathrm{Re}\, a > 0$ and $\mathrm{Re}\, z > 0$. The factor $1/\Gamma(a)$ is not essential for our analysis, nevertheless, we include it to keep our notation in agreement with the standard one. The role of this factor is to normalize the leading term in the asymptotic $U(a,b,z) = z^{-a}(1+\mathrm{O}(z^{-1}))$ as $z \to \infty$. It is easy to check by direct calculation that the function U given by (2.4) solves the equation (2.2). We have

$$\frac{d^k}{dz^k} U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^\infty (-t)^k e^{-tz} t^{a-1} (1+t)^{b-a-1}\, dt\,,$$

and after substituting to (2.2) we see from a simple integration by parts that the equation is satisfied.

By letting $zt = s$ in (2.4) we obtain (for $\mathrm{Re}\, z > 0$, $\mathrm{Re}\, a > 0$)

$$U(a,b,z) = \frac{1}{\Gamma(a)} z^{-a} \int_0^\infty e^{-s} s^{a-1} \left(1 + \frac{s}{z}\right)^{b-a-1} ds\,. \tag{2.5}$$

The above expression is used to extend the definition of U for $\mathrm{Re}\, a > 0$ and $z \in \mathbb{C}$ with $-\pi < \arg z < \pi$, since the integral is convergent and an analytic function of z under these assumptions. By formally expanding the term $(1+s/z)^{b-a-1}$ and integrating the resulting (formal) series term-by-term, we obtain the following asymptotic expansion

$$U(a,b,z) = z^{-a} \left(\sum_{k=0}^{n-1} \frac{(a)_k (1+a-b)_k}{k!} (-z)^{-k} + \mathrm{O}(|z|^{-n}) \right)\,, \tag{2.6}$$

where $(a)_0 = 1$, $(a)_k = a(a+1)\dots(a+k-1)$. The formal calculations of the asymptotic expansion (2.6) can be easily justified rigorously by splitting the integral in (2.5) as $\int_0^\infty = \int_0^{s_1} + \int_{s_1}^\infty$ with $s_1 = |z|^{1/2}$, for example.

One can check easily by a direct calculation that the function

$$V(a,b,z) = e^z U(b-a, b, -z) \tag{2.7}$$

is another solution of Kummer's equation and that the functions U and V are linearly independent. The Wronskian of U and V is given by

$$U \frac{\mathrm{d}V}{\mathrm{d}z} - V \frac{\mathrm{d}U}{\mathrm{d}z} = e^{\pm i\pi(b-a)} z^{-b} e^z\,, \tag{2.8}$$

where the sign $+$ is for $\mathrm{Im}\, z > 0$ and $-$ in the opposite case. (Formula (2.8) is easily derived from the fact that the Wronskian satisfies the differential equation $y' + (b/z - 1)y = 0$ and from the asymptotic expansion (2.6).)

We need to know the behavior of U and V in the region $-\pi/2 \le \arg z < 0$. Taking into account the definition of V, we see that it is sufficient to control U in the region $-\pi/2 \le \arg z < \pi$. Formula (2.5) is suitable for analysis in this region if $\mathrm{Re}\,(b-a) > 0$ since the integral in (2.5) is the uniformly absolutely convergent whenever z approaches a point in $(-\infty, 0)$ from the upper half-plane. In our applications the condition $\mathrm{Re}\,(b-a) > 0$ is always satisfied and therefore (2.5) is sufficient for our analysis.

We now have sufficient information about the solutions of (2.2), and hence also (2.1), to be able to proceed with the analysis of the inhomogeneous equation

$$(1-i\epsilon)u'' + (1-i\epsilon)\frac{d-1}{\xi}u' + i\kappa\xi u' + \frac{i\kappa}{\sigma}u - \omega u = f(\xi)\,, \tag{2.9}$$

for $\xi \in (\xi_1, \infty)$. We assume that the function f is decaying sufficiently fast as $\xi \to \infty$. We are interested in solutions of (2.9) which have the asymptotics $u \sim \xi^{-1/\sigma - i\omega/\kappa}$ as $\xi \to \infty$. We denote P, E two linearly independent solutions of (1.4)

$$\begin{aligned}
&P(\xi) \equiv P(\kappa,\omega,\epsilon;\xi) = U(a,b,z)\,,\\
&E(\xi) \equiv E(\kappa,\omega,\epsilon;\xi) = V(a,b,z)\,,\\
&\text{where }\ z = \frac{-i\kappa}{1-i\epsilon}\frac{\xi^2}{2}\,,\ a = \frac{1}{2}\left(\frac{1}{\sigma} + i\frac{\omega}{\kappa}\right)\,,\ b = \frac{d}{2}\,.
\end{aligned}$$

The parameters d and σ are held fixed in the perturbation analysis described below, therefore we do not indicate the dependence of P and E on them. The Wronskian $W = PE' - P'E$ is easily computed from (2.8).

$$W(\xi) = W(\omega,\kappa,\epsilon;\xi) = \frac{-i\kappa}{1-i\epsilon}e^{\pm\pi i(b-a)}\xi z^{-b}e^{z}\,. \tag{2.10}$$

The next step is to use the standard variation of constant to obtain solutions to (2.9) satisfying $u(\xi) \sim \xi^{-1/\sigma - i\omega/\kappa}$ as $\xi \to \infty$. We write the solution in the form

$$u(\xi) = c_1(\xi)P(\xi) + c_2(\xi)E(\xi)\,,\ \text{with} \tag{2.11}$$

$$u'(\xi) = c_1(\xi)P'(\xi) + c_2(\xi)E'(\xi)\,, \tag{2.12}$$

and $c_2(\xi) \to 0$ "sufficiently fast" as $\xi \to \infty$. We obtain

$$c_1'(\xi) = -f(\xi)\frac{E(\xi)}{(1-i\epsilon)W(\xi)}\,,\quad c_2'(\xi) = f(\xi)\frac{P(\xi)}{(1-i\epsilon)W(\xi)}\,, \tag{2.13}$$

which together with the condition $c_2(\xi) \to 0$ at the infinity gives a formal expression for the solution

$$u(\xi) = \gamma P(\xi) + \int_{\xi_1}^{\infty} K(\xi,\eta)f(\eta)\,d\eta\,, \tag{2.14}$$

where $\gamma \in \mathbb{C}$ is a constant and

$$K(\xi,\eta) = \left\{\begin{array}{ll} -\frac{1}{(1-i\epsilon)}P(\xi)E(\eta)W^{-1}(\eta) & \text{for } \xi_1 < \eta \le \xi \\ -\frac{1}{(1-i\epsilon)}E(\xi)P(\eta)W^{-1}(\eta) & \text{for } \xi \le \eta \end{array}\right.\,.$$

The strategy for finding solutions of the non-linear problem (1.4) in (ξ_1, ∞) with the required decay $u(\xi) \sim \xi^{-1/\sigma - i\omega/\kappa}$ at infinity should now be clear. Heuristically we expect that the manifold of such solutions will be a deformation of the one-dimensional complex subspace $\{\gamma P \,|\, \gamma \in \mathbb{C}\}$, at least in a neighborhood of the origin. The deformation is from γP to the fixed point of the operator T

$$u \mapsto Tu(\xi) = \gamma P(\xi) - \int_{\xi_1}^{\infty} (1+i\delta)K(\xi,\eta)\,|u(\eta)|^{2\sigma}u(\eta)\,d\eta\,.$$

The outlined strategy can be successfully carried out by using the properties of Kummer's functions recalled above. The fixed point theorem can be applied in the Banach space

$$\mathcal{X}_\vartheta = \{u \in C([\xi_1, \infty)) \mid \sup_{\xi \geq \xi_1} |\xi|^{1/\sigma - \vartheta} |u(\xi)| < \infty\}$$

equipped with the norm

$$\|u\|_\vartheta = \sup_{\xi \geq \xi_1} |\xi|^{1/\sigma - \vartheta} |u(\xi)| \,.$$

This approach is obviously standard. However, there are some subtle points in the situation studied here due to oscillatory behavior of the function E, see the Appendix.

The main result of this section, which can be derived in a fully rigorous way from the above analysis is the following theorem. (A complete proof of the theorem is presented in the Appendix.)

THEOREM 2.1. *Assume* $1 \leq d \leq 3$, $2/d < \sigma < 2/d + 1/2$, $0 < \kappa_1 < \kappa_2$, $\omega_1 < \omega_2$, $0 < \epsilon_1$, $\delta_1 < \delta_2$.

There exists $\rho_1 > 0$ *such that for each* $\xi_1 \geq 1$ *and each*

$$(\beta, \kappa, \omega, \epsilon, \delta) \in \Lambda_{\rho_1} \equiv \{\beta \in \mathbb{C} \,|\, |\beta| \leq \rho_1\} \times [\kappa_1, \kappa_2] \times [\omega_1, \omega_2] \times [0, \epsilon_1] \times [\delta_1, \delta_2]$$

the boundary value problem

$$(1 - i\epsilon)(Q'' + \frac{d-1}{\xi} Q') + i\kappa\xi Q' + i\frac{\kappa}{\sigma} Q - \omega Q + (1 + i\delta)|Q|^{2\sigma} Q = 0 \,, \qquad (2.15)$$

$$Q(\xi_1) = \beta \,, \quad \textit{and} \qquad (2.16)$$

$$Q(\xi) \sim \xi^{-1/\sigma - i\omega/\kappa} \quad \textit{as} \;\; \xi \to \infty \,. \qquad (2.17)$$

considered in (ξ_1, ∞) *has a solution*

$$F(\xi) = F(\beta, \kappa, \omega, \epsilon, \delta, \xi_1; \xi) \,.$$

Moreover, F can be constructed in such a way that the following conditions are satisfied

(i) *The mapping from* Λ_{ρ_1} *to* $\mathcal{X}_\vartheta$ *defined by*

$$(\beta, \kappa, \omega, \epsilon, \delta) \mapsto F(\beta, \kappa, \omega, \epsilon, \delta, \xi_1; \cdot)$$

is C^1 up to the boundary for each $\vartheta > 0$.

(ii) *The complex-valued function defined by*

$$(\beta, \kappa, \omega, \epsilon, \delta) \mapsto \frac{\partial F}{\partial \xi}(\beta, \kappa, \omega, \epsilon, \delta, \xi_1, \xi_1)$$

is C^1 (up to the boundary) in Λ_{ρ_1}.

(iii) *F and its derivatives $F^{(k)}$ have the following asymptotic expansions*

$$F(\xi) = \xi^{-1/\sigma - i\omega/\kappa} \left(\sum_{l=0}^{n} a_l \xi^{-2l} \; + \mathrm{O}(\xi^{-2(n+1)}) \right) \,, \qquad (2.18)$$

$$F^{(k)}(\xi) = \frac{\partial^k}{\partial \xi^k} \left(\xi^{-1/\sigma - i\omega/\kappa} \sum_{l=0}^{n} a_l \xi^{-2l} \right) \; + \mathrm{O}(\xi^{-1/\sigma - 2(n+1) - k}) \,. \quad (2.19)$$

(iv) *We have* $F(0,\kappa,\omega,\epsilon,\delta,\xi_1,\xi)=0$ *and*

$$\frac{\partial F}{\partial \beta}(0,\kappa,\omega,\epsilon,\delta,\xi_1,\xi)=\frac{P(\kappa,\omega,\epsilon,\xi)}{P(\kappa,\omega,\epsilon,\xi_1)}\,.$$

Proof: See the Appendix

Remark 2.1. *For $\epsilon>0$ the function $u(x,t)$ given by (1.3) has to be regular at all points (x,t) with $x\neq 0$ by the partial regularity theorem proved in [17]. Therefore any solution of (1.4)-(1.6) must admit an asymptotic expansion of the form stated in (iii) of Theorem 2.1. We note that the convergence of the series $\sum a_l\xi^{-2l}$ in the asymptotic expansion of Q is equivalent to the analycity of u (in t) at the points (x,T), $x\neq 0$. The asymptotic expansion of the profiles Q does not converge and therefore u is not analytic in t at any point (x,T).*

Theorem 2.1 together with elementary perturbation arguments can be used to show that if some non-degeneracy conditions are satisfied, then every solution of (1.4)-(1.6) for $\epsilon=\delta=0$ will persist (with a slight deformation) for small $\epsilon>0$ and $\delta>0$. We will briefly describe this standard procedure for the convenience of the reader.

First we consider solutions on the finite interval $(0,\xi_1]$ to the initial value problem

$$(1-i\epsilon)(Q''+\frac{d-1}{\xi}Q')+i\kappa\xi Q'+i\frac{\kappa}{\sigma}Q-\omega Q+(1+i\delta)|Q|^{2\sigma}Q=0\,, \tag{2.20}$$

$$Q(0)=\mu\,,\quad Q'(0)=0\,. \tag{2.21}$$

We denote the solution (if it exists) by $G(\xi)=G(\mu,\kappa,\omega,\epsilon,\delta;\xi)$. Clearly the set of parameters $(\mu,\kappa,\omega,\epsilon,\delta)$ for which G is well-defined is open. We define

$$\beta(\mu,\kappa,\omega,\epsilon,\delta)=G(\mu,\kappa,\omega,\epsilon,\delta;\xi_1)\,. \tag{2.22}$$

Assume that the boundary-value problem (1.4)-(1.6) has a solution which satisfies $Q(\xi_1)=\mu$. With a slight abuse of notation, let us denote such a solution by $Q(\mu,\kappa,\omega,\epsilon,\delta;\xi)$. Clearly $Q(\mu,\kappa,\omega,\epsilon,\delta;\xi)$ is defined only on a submanifold of the parameter space, but this will not be important in what follows. As we have seen in the introduction, we have

$$\lambda^{1/\sigma+i\theta}Q(\mu,\kappa,\omega,\epsilon,\delta,\lambda\xi)=Q(\lambda^{1/\sigma+i\theta}\mu,\lambda^2\kappa,\lambda^2\omega,\epsilon,\delta;\xi)\,,$$

for all $\lambda>0$ and $\theta\in[0,2\pi)$. Therefore we can work with normalized solutions, i. e. we assume that $\omega=1$ and that $Q(0)$ is real and non-negative. Assume $\mu_0>0,\kappa_0>0$ and suppose $Q(\mu_0,\kappa_0,1,0,0;\xi)$ exists.

We set

$$g(\mu,\kappa,\epsilon,\delta)=\frac{\partial G}{\partial \xi}(\mu,\kappa,1,\epsilon,\delta;\xi_1)-\frac{\partial F}{\partial \xi}(\beta(\mu,\kappa,1,\epsilon,\delta),\kappa,1,\epsilon,\delta,\xi_1,\xi_1)\,. \tag{2.23}$$

where $\beta(\mu,\kappa,\omega,\epsilon,\delta)$ is defined by (2.22). By our assumptions and by Theorem 2.1 the mapping g is well defined and continuously differentiable in a set of the form $(\mu_1,\mu_2)\times(\omega_1,\omega_2)\times[0,\epsilon_1)\times(\delta_1,\delta_2)$ containing the point $(\mu_0,\kappa_0,0,0)$. Since $g(\mu_0,\kappa_0,0,0)=0$, we see that the equation

$$g(\mu,\kappa,\epsilon,\delta)=0$$

has solutions for small $\epsilon > 0$, $\delta > 0$ if the following non-degeneracy condition is satisfied:

$$\det \begin{pmatrix} g^1_\mu & g^1_\kappa \\ g^2_\mu & g^2_\kappa \end{pmatrix} \neq 0 \quad \text{at} \quad (\mu_0, \kappa_0, 0, 0)\,, \tag{2.24}$$

where $g^1 = \operatorname{Re} g$, $g^2 = \operatorname{Im} g$ and subscripts denote partial derivatives with respect to the corresponding variables.

Thus a non-trivial solution of (1.4)-(1.6) for $\epsilon = \delta = 0$ also gives a solution of (1.4)-(1.6) for $\epsilon, \delta > 0$ if (2.24) is satisfied. Based on our numerical calculations described in the next section, we conjecture that (2.24) is satisfied for $\sigma > 2/d$.

3. Numerical results. Theorem 2.1 allows us to rewrite the boundary value problem (1.4)-(1.6) as a boundary value problem on a finite interval $(0, \xi_1)$ in the following way.

$$(1 - i\epsilon)(Q'' + \frac{d-1}{\xi} Q') + i\kappa\xi Q' + i\frac{\kappa}{\sigma} Q - Q + (1 + i\delta)|Q|^{2\sigma} Q = 0\,, \tag{3.1}$$

$$Q'(0) = 0\,, \tag{3.2}$$

$$Q(\xi_1) = \beta\,, \tag{3.3}$$

$$Q'(\xi_1) = \frac{\partial F}{\partial \xi}(\beta, \kappa, 1, \epsilon, \delta, \xi_1; \xi_1)\,, \tag{3.4}$$

where the unknown quantities are Q, β and κ. Of course, the problem (3.1)-(3.4) is equivalent to the equation $g = 0$ in the previous section. If we approximate F by the first few terms of its asymptotic expansion, the problem (3.1)-(3.4) can be solved numerically. In our numerical computations we investigated the dependence on ξ_1 and on the number of terms of the asymptotic expansion of F. It turned out that $\xi_1 \sim 30$ and the first term of the asymptotic expansion already worked very well. However, many of our computations were done with the first two terms of the asymptotic expansion (2.18). In the case $\epsilon = \delta = 0$, i.e. NLS, the value of (μ_1, κ_1) was computed in [7] using completely different approach. The values presented in that paper are in an excellent agreement with our computations, see below.

The first term of the asymptotic expansion of F is $F \sim \beta(\xi/\xi_1)^{-1/\sigma - i\omega/\kappa}$. Using this approximation, we obtain from (3.1)-(3.4)

$$(1 - i\epsilon)(Q'' + \frac{d-1}{\xi} Q') + i\kappa\xi Q' + i\frac{\kappa}{\sigma} Q - Q + (1 + i\delta)|Q|^{2\sigma} Q = 0\,, \tag{3.5}$$

$$Q'(0) = 0\,, \tag{3.6}$$

$$\xi_1 Q'(\xi_1) + \left(\frac{1}{\sigma} + i\frac{1}{\kappa}\right) Q(\xi_1) = 0\,. \tag{3.7}$$

Higher order approximations can be derived in a similar way. Note that in the formulation of (3.5) we already fixed the normalization $\omega = 1$, so that the unknowns are Q and κ. The boundary condition (3.7) is also closely related to the boundary condition used in [4], [5] for simulations based on solving time dependent problem in the PDE (1.1)-(1.2).

There are essentially two approaches to the numerical solution of the boundary-value problem (3.5)-(3.7). One can use collocation methods to approximate the boundary-value problem and then to apply Newton's method to the discretization

of the non-linear operator which defines the equation (3.5). Implementation of this strategy requires further changes in the formulation since the linearization of the non-linear operator always has zero in its spectrum due to the S^1-equivariance of the equation. Therefore a shooting method was easier to implement and it also proved to be sufficiently accurate. Because of well-known sensitivity of shooting methods to problem parameters we performed computations in different normalizations: with $Q(0) = 1$ fixed and parameters (κ, ω) as the unknowns as well as with $\omega = 1$ fixed and parameters (κ, μ) as the unknowns. Moreover we compared both backward and forward shooting methods on the interval $(0, \xi_1)$. All computed solutions turned out to be in a very good agreement.

Here we describe only the shooting method using the normalization $\omega = 1$, in which we calculate $\xi_1 Q'(\xi_1) + (1/\sigma + i/\kappa)Q(\xi_1)$ as a function of the parameters $\mu = Q(0), \kappa, \epsilon$, and δ, i.e., we solve the equation

$$f(\mu, \kappa, \epsilon, \delta) \equiv \xi_1 Q'(\xi_1) + (1/\sigma + i/\kappa)\, Q(\xi_1) = 0\,. \tag{3.8}$$

The integration of the underlying ODE must be done with sufficient accuracy. We compared various ODE solvers. A variable-order, variable-step Adams method as implemented, for example, in the NAG library proved to be sufficiently accurate in most of the calculations. To locate the initial values for each branch we inspected the two dimensional subspace of the parameter space given by $(\mu, \kappa, 0, 0)$ for $d = 1$ and $\sigma = 2.3$ and computed the degree of the function f restricted to that subspace along various curves. The other solutions were then calculated by continuation. This was done with the help of bifurcation analysis package developed as a part of [12]. The implementation of the path-following procedure with a Newton corrector step can be done efficiently as the linearization along a solution is evaluated at the same integration step as the solution.

As we described in the introduction, it appears that the equation $f(\mu, \kappa, 0, 0)$ has countably many solutions (κ_j, μ_j) in the region $\kappa > 0$, $\mu > 0$. The corresponding profiles $|Q_j|$, when extended to $(-\infty, \infty)$ as even functions, have exactly j local maxima.

Remark 3.1. *As mentioned in the beginning of this section the solutions (κ_j, μ_j) may slightly depend on the value ξ_1 and the approximation of F. For solutions described here we tested the dependence of the results on ξ_1 for $\xi_1 \in [20, 100]$, and also on the approximation of F by taking either one or two terms in the asymptotics expansion of F. We also directly compared our numerical solutions in intervals of the form (η_1, ξ_1) with the explicit formulae given by one or two terms of the asymptotic expansion of the solution in (η_1, ξ_1) for various values of η_1. All these tests indicated a good convergence of our approximations. Based on these tests, we estimate that the error in the values of the "roots" (κ_j, μ_j) is of the order 10^{-3} or better.*

Case I ($d = 1$, $\sigma = 2.3$, $\delta = 0$): Results for the one-dimensional case are tabulated in Table 3.1.

The continuation of solutions parameterized by ϵ is depicted in Figure 3.1. To give the reader a good idea about the form of the solutions we plot profiles $|Q(\xi)|$ at a few points on each branch (at the point $\epsilon = 0$ (solution to NLS), at a point on the upper part of the branch and at another point on the lower part of the branch). We used both normalizations, $Q(0) = 1$ in Figure 3.2–3.3, and $\omega = 1$ in Figure 3.4–3.5. Normalization $Q(0) = 1$ is convenient for computing solutions along the odd branches as these solutions exhibit a maximum at the origin which is also present in the solution for $\epsilon \to 0$ on the lower part of the branch.

branch	turning point	$\omega = 1$		$Q(0) = 1$	
j	ϵ^*	κ	μ	κ	ω
1	0.06064	0.85311	1.23204	0.32669	0.38294
2	0.05182	0.49323	0.78308	1.51894	3.07959
3	0.04466	0.34673	1.12388	0.20263	0.58438
4	0.03900	0.26678	0.78308	0.47127	1.76651
5	0.03455	0.21643	1.07947	0.15225	0.70345
6	0.03099	0.18185	0.92714	0.25750	1.41624
7	0.02803	0.15667	1.05430	0.12284	0.78409
8	0.02559	0.13756	0.95061	0.17365	1.26236

TABLE 3.1

CASE I *($d = 1$, $\sigma = 2.3$, $\delta = 0$): ϵ^* defines the position of the turning point; coordinates (κ, μ), where $\mu = Q(0)$ define the initial point of each branch, when $\epsilon = 0$, in the normalization $\omega = 1$. The coordinates (κ, ω) refer to the same solutions in the normalization $Q(0) = 1$.*

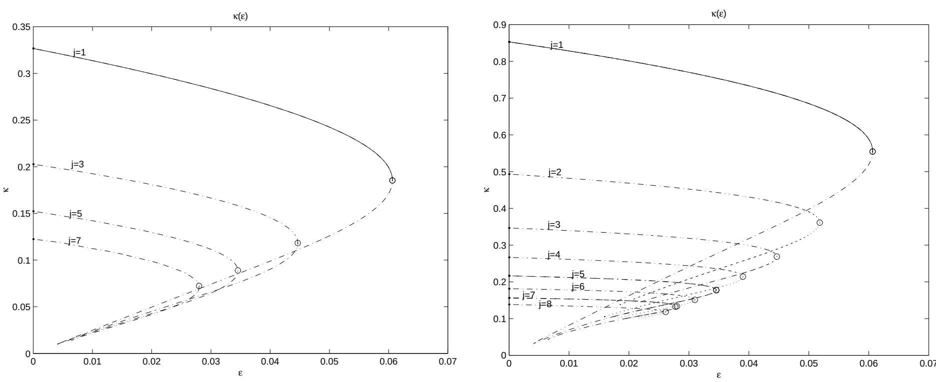

FIG. 3.1. CASE I *($d = 1$, $\sigma = 2.3$, $\delta = 0$): solution branches $(\epsilon, \kappa_j(\epsilon))$; normalization $Q(0) = 1$ (left) and $\omega = 1$ (right). Turning points are denoted by o. The solid line indicates stable solutions, the dashed line indicates unstable solutions. See Section 4 for details concerning stability issues.*

We now describe the behavior of solutions along the branches. We observe that as ϵ returns to zero after passing through the turning point, we have $\kappa_j(\epsilon) \to 0$.

For $j = 1$ the solution $Q_1^\epsilon(\xi)$ approaches a specific solution, usually called *the ground state*, of the equation

$$-u'' + u = |u|^{2\sigma} u\,. \tag{3.9}$$

For $j > 1$ the profile $|Q_j^\epsilon(\xi)|$ seems to separate into j copies of the ground-state solution which are moving to infinity. For j even all of them “escape” to infinity, while for j odd one will stay at the origin and the rest will move to the infinity as $\epsilon \to 0$ and $\kappa_j(\epsilon) \to 0$.

Case II ($d = 3$, $\sigma = 1$, $\delta = 0$): Results for the three-dimensional case are tabulated in Table 3.2.

Some data for NLS ($\epsilon = \delta = 0$) and the “basic solution” (corresponding to the beginning of our first branch) are available in the literature and can be used to estimate accuracy of our calculations. One can see a very good agreement of our solution for $j = 1$ with values $\kappa = 0.917$ and $\mu \equiv Q(0) = 1.885$ obtained in [7] from simulations that used the dynamical rescaling method applied to the initial value problem (1.1)-(1.2).

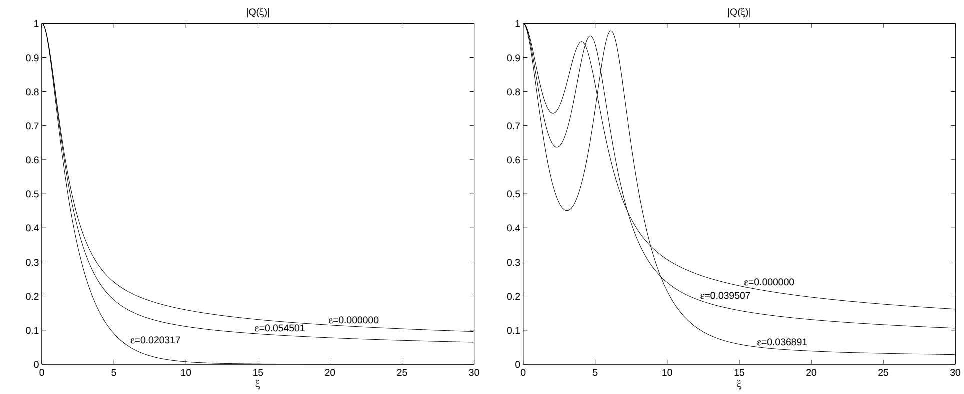

FIG. 3.2. CASE I *($d = 1$, $\sigma = 2.3$, $\delta = 0$): profiles $|Q(\xi)|$ at points along the j-th branch; $j = 1$ (left), $j = 3$ (right); normalization $Q(0) = 1$.*

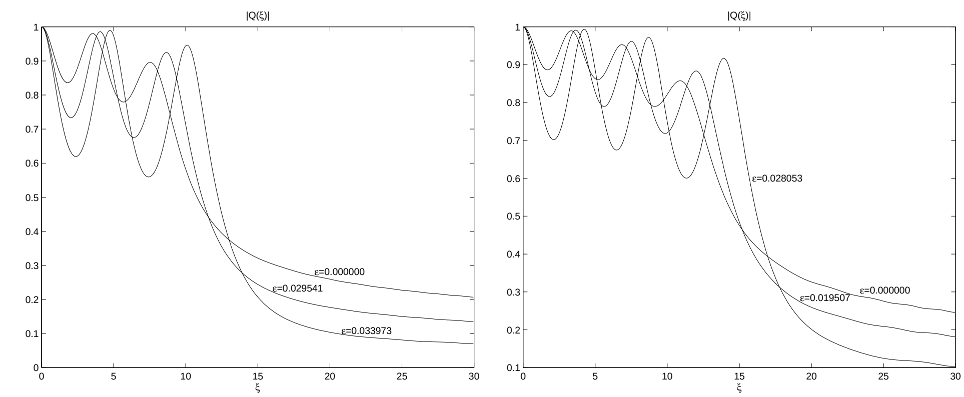

FIG. 3.3. CASE I *($d = 1$, $\sigma = 2.3$, $\delta = 0$): profiles $|Q(\xi)|$ at points along the j-th branch; $j = 5$ (left), $j = 7$ (right); normalization $Q(0) = 1$.*

The behavior of solutions Q_j^ϵ when ϵ returns to zero is qualitatively similar to the one-dimensional case, but there are also some new interesting features.

First, the maximal possible ϵ_j^* is not attained on the first branch ($j = 1$). Second, the behavior as $\epsilon \to 0$ and $\kappa_j(\epsilon) \to 0$ is different for even and odd values j. For $j = 2k - 1$ (odd) the solutions $Q_j^\epsilon(\xi)$ converge (along the jth-branch) to the k-th radial solution of the problem

$$- \Delta u + u = |u|^{2\sigma} u \ \text{ in } \ \mathbb{R}^3 \,, \tag{3.10}$$

$$u \to 0 \ \text{ as } \ |x| \to \infty \,. \tag{3.11}$$

For $j = 2k$ (even), one observes a similar behavior as in the one-dimensional case.

REMARK 3.2. *From the regularity theory for CGL (see [1]) one expects that the value of ϵ cannot cross a certain finite value $\tilde{\epsilon}$, as we follow solutions along any branch. What we see in the calculations is a turning point on each branch. In Table 3.1 and Table 3.2 we list for each of the calculated branches the parameter ϵ^* which is the maximal value of the perturbation parameter ϵ reached along the branch.*

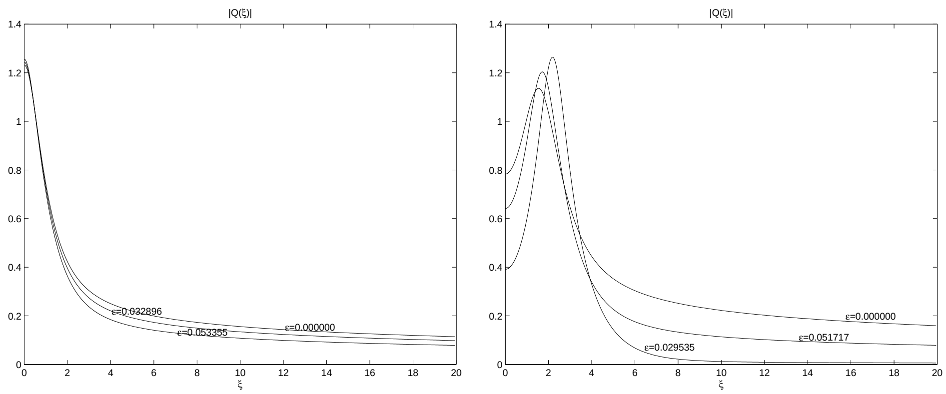

FIG. 3.4. CASE I *($d = 1$, $\sigma = 2.3$, $\delta = 0$): profiles $|Q(\xi)|$ at points along the j-th branch; $j = 1$ (left), $j = 2$ (right); normalization $\omega = 1$.*

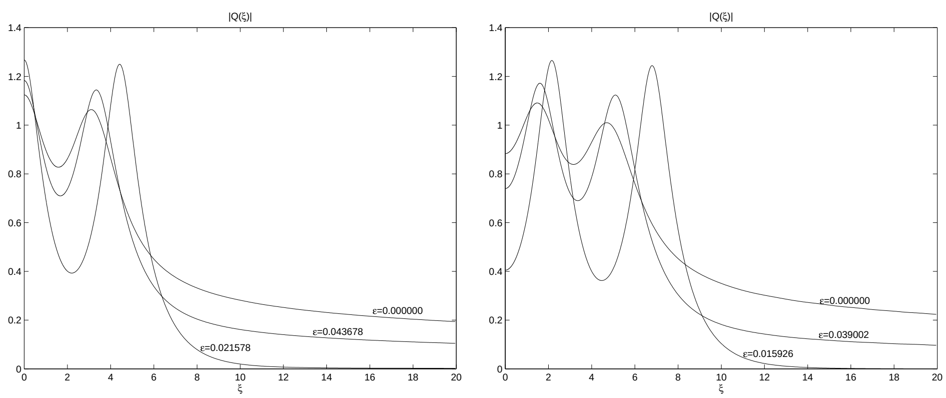

FIG. 3.5. CASE I *($d = 1$, $\sigma = 2.3$, $\delta = 0$): profiles $|Q(\xi)|$ at points along the j-th branch; $j = 3$ (left), $j = 4$ (right); normalization $\omega = 1$.*

4. Stability of singular solutions. A natural question regarding the stability of the self-similar singularities constructed in the previous section is for example the following: What will be the behavior of solutions to (1.1) and (1.2) if u_0 is a slight perturbation of a solution Q of the profile equation (1.4)-(1.6) and f is small? (One can think for example of taking $u_0 = \varphi Q + u_1$, where φ is a compactly supported smooth cut-off function which is identically one in a large ball, Q is a non-trivial solution of (1.4)-(1.6), u_1 is small and compactly supported, and at the same time taking f which is small and compactly supported in x.)

Here we present an approach to this problem which uses the method of dynamical rescaling (see for example [10]), which seems to be natural in this context. We emphasize that we will not obtain fully rigorous analytical results which would completely answer the question raised above. Our goal is to present some preliminary calculations which seem to be adequate to the issue of stability in the context of numerical simulations.

We briefly recall the main idea of the method of dynamical rescaling. We consider

branch	turning point	$\omega = 1$		$Q(0) = 1$	
j	ϵ^*	κ	μ	κ	ω
1	0.19813	0.91737	1.88529	0.25810	0.28135
2	0.24402	0.32091	0.83559	0.45535	1.41727
3	0.22762	0.22704	1.10834	0.18242	0.80684
4	0.19520	0.16543	1.03257	0.15516	0.93792
5	0.18168	0.14237	1.00325	0.13241	0.96677

Table 3.2

Case II *($d = 3$, $\sigma = 1$, $\delta = 0$): ϵ^* defines the position of the turning point; coordinates (κ, μ), where $\mu = Q(0)$ define the initial point of each branch, when $\epsilon = 0$, in the normalization $\omega = 1$. The coordinates (κ, ω) refer to the same solutions in the normalization $Q(0) = 1$. In [7] the solution corresponding to $j = 1$ and $\epsilon = 0$ was calculated by the method of dynamical rescaling, which gave $\kappa = 0.917$ and $\mu = 1.885$.*

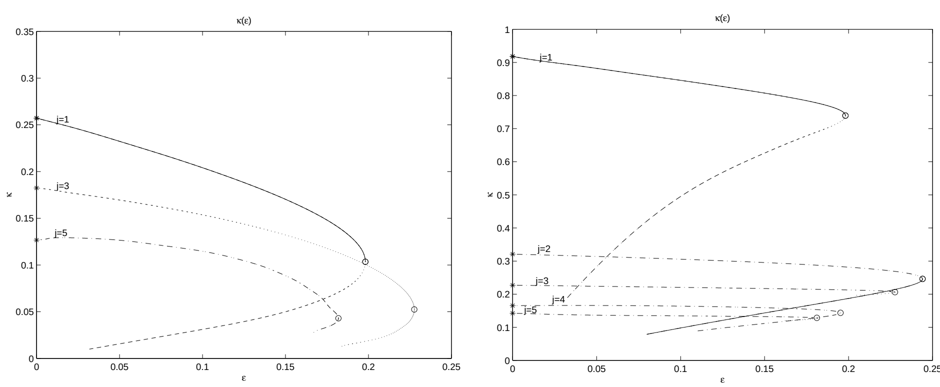

Fig. 3.6. Case II *($d = 3$, $\sigma = 1$, $\delta = 0$): solution branches $(\epsilon, \kappa_j(\epsilon))$; normalization $Q(0) = 1$ (left) and $\omega = 1$ (right). Turning points are denoted denoted by o. The solid line indicates stable solutions, the dashed line indicates unstable solutions. See Section 4 for details regarding stability issues.*

a solution $u : \mathbb{R}^d \times (0, T) \to \mathbb{C}$ of the complex Ginzburg-Landau equation

$$i\frac{\partial u}{\partial t} + (1 - i\epsilon)\Delta u + (1 + i\delta)|u|^{2\sigma}u = 0\,. \tag{4.1}$$

Motivated by the scaling invariance of solutions

$$u(x, t) \to \lambda^{1/\sigma} u(\lambda x, \lambda^2 t)$$

of (4.1), we write the solution u as

$$u(x, t) = L^{-1/\sigma}(t)\, v\left(L^{-1}(t)x, \tau\right)\,,$$

where $L(t) > 0$ is to be chosen and $d\tau = L^{-2}(t)\,dt$. From (4.1) we obtain

$$i\frac{\partial v}{\partial \tau} + i\kappa(\tau)\left(\xi\frac{\partial v}{\partial \xi} + \frac{1}{\sigma}v\right) + (1 - i\epsilon)\Delta v + (1 + i\delta)|v|^{2\sigma}v = 0\,, \tag{4.2}$$

where $\kappa(\tau) = -L^{-1}(\tau)\,dL(\tau)/\,d\tau = -L(t)\,dL(t)/\,dt$.

We now chose $L(t)$ so that, roughly speaking, the typical length-scale over which v oscillates is 1. If u develops a singularity at $t = T$ (and it is regular in $(0, T_1)$ for each $T_1 < T$), then $L(t) > 0$ in $(0, T)$ and $L(t) \to 0$ as $t \to T$.

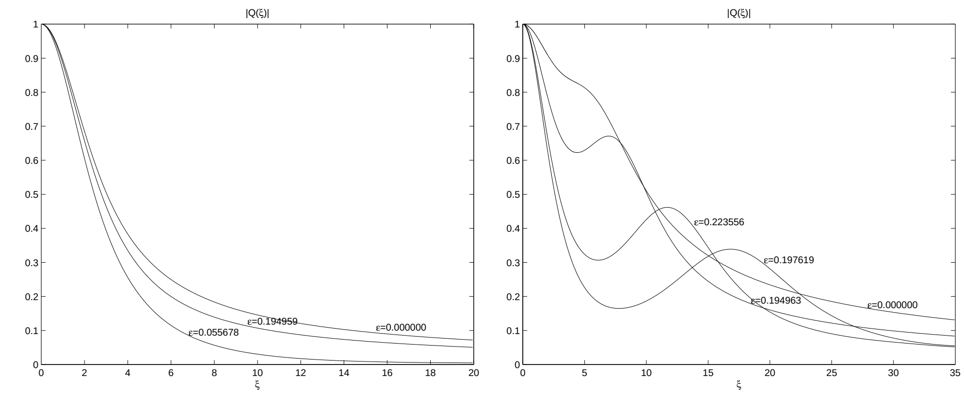

FIG. 3.7. CASE II *($d = 3$, $\sigma = 1$, $\delta = 0$): the profile $|Q(\xi)|$ at points along the j-th branch; $j = 1$ (left), $j = 3$ (right); normalization $Q(0) = 1$.*

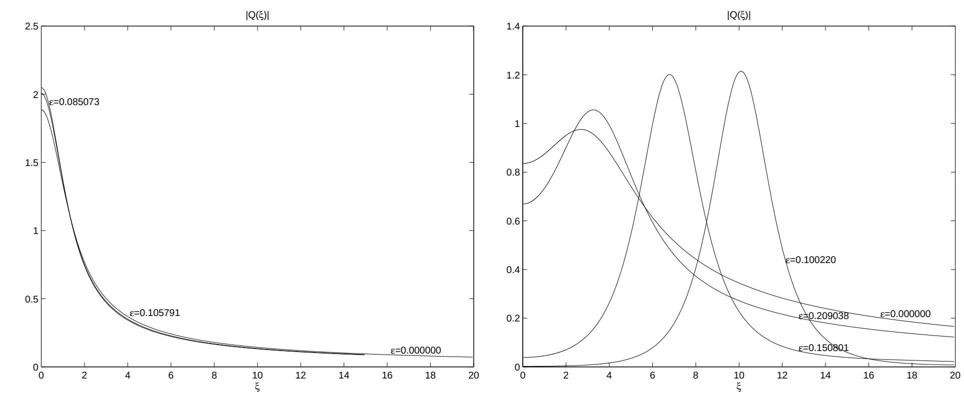

FIG. 3.8. CASE II *($d = 3$, $\sigma = 1$, $\delta = 0$): the profile $|Q(\xi)|$ at points along the j-th branch. $j = 1$ (left), $j = 2$ (right); normalization $\omega = 1$.*

One way to control oscillations of v is to impose a condition $\mathcal{J}(v(.,\tau)) = 1$, where $\mathcal{J}$ is a suitable functional controlling regularity of solutions to (4.2). One is then led to the following system

$$i\frac{\partial v}{\partial \tau} + i\kappa(\tau)\left(\xi\frac{\partial v}{\partial \xi} + \frac{1}{\sigma}v\right) + (1 - i\epsilon)\Delta v + (1 + i\delta)|v|^{2\sigma}v = 0\,, \tag{4.3}$$

$$\mathcal{J}(v(.,\tau)) = 1\,, \tag{4.4}$$

where the unknowns are v and $\kappa(\tau)$. Various choices of the functional $\mathcal{J}$ have been used for numerical calculations. For example, in [10, 6] the functional

$$\mathcal{J}_0(v) = \int_{\mathbb{R}^d} |\nabla v|^2\, dx$$

was used. Although it does not directly follow from the known regularity theory that $\mathcal{J}_0$ controls the regularity of solutions to (4.2) in the case $\epsilon = \delta = 0$ this choice turned out to work satisfactorily in the numerical computations.

In the following analysis we assume that the functional $\mathcal{J}$ is invariant under the action of the symmetry group S^1, i.e., $\mathcal{J}(e^{i\theta}v) = \mathcal{J}(v)$ for all $\theta \in [0, 2\pi)$. Self-similar singularities of the form (1.3) correspond to solutions of (4.3) of the form

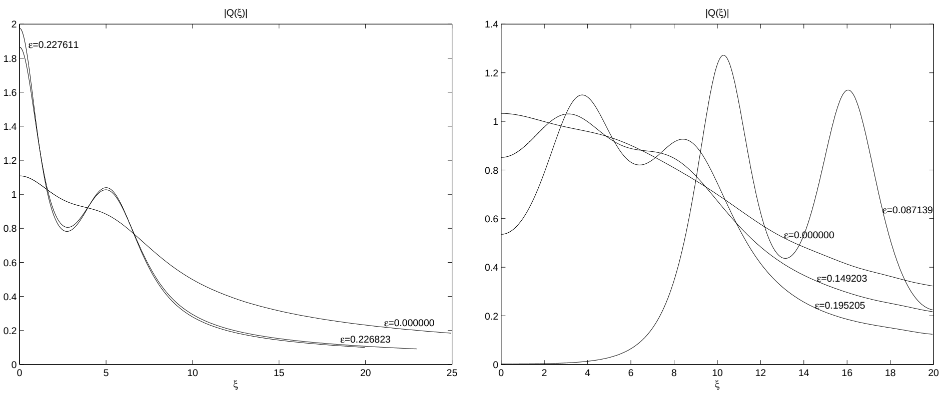

FIG. 3.9. CASE II *($d = 3$, $\sigma = 1$, $\delta = 0$): the profile $|Q(\xi)|$ at points along the j-th branch $j = 3$ (left), $j = 4$ (right); normalization $\omega = 1$.*

$v(\xi,\tau) = e^{i\omega\tau}Q(\xi)$, i.e., to S^1-orbits of solutions to the problem (1.4)-(1.6). We will consider the linearized stability of these orbits, which seems to be the simplest natural notion of stability in our context. Suppose $(v_0(\xi,\tau),\kappa_0(\tau))$ is a solution of (4.3) such that $v_0(\xi,\tau) = e^{i\omega_0\tau}Q(\xi)$ and $\kappa_0(\tau) = \kappa_0 = \text{const}$. We consider a perturbed solution $(v(\xi,\tau),\kappa(\tau))$ in the form

$$\begin{aligned} v(\xi,\tau) &= e^{i\omega_0\tau}(Q(\xi) + w(\xi,\tau)) \\ \kappa(\tau) &= \kappa_0 + \varkappa(\tau)\,, \end{aligned}$$

where w and $\varkappa$ are infinitesimally small. A simple calculation gives

$$\frac{\partial w}{\partial \tau} = \mathcal{L}w + \varkappa(\tau)\left(\xi Q' + \frac{1}{\sigma}Q\right), \tag{4.5}$$

$$\langle \mathcal{J}'(Q), w\rangle = 0\,, \tag{4.6}$$

where the operator $\mathcal{L}$ is defined as the linearization of the equation (4.2) along the solution (v_0,κ_0), i.e.,

$$\mathcal{L}w = (i+\epsilon)\Delta w - \kappa_0\left(\xi\frac{\partial w}{\partial \xi} + \frac{1}{\sigma}w\right) - i\omega_0 w + i(1+\sigma)|Q|^{2\sigma}w + i\sigma|Q|^{2\sigma-2}Q^2\bar{w}$$

and $\mathcal{J}'$ denotes the derivative of $\mathcal{J}$. (Note that the operator $\mathcal{L}$ is not complex linear and therefore it is natural to carry out the analysis in the real representation.)

There are two eigenfunctions of $\mathcal{L}$ that can be formally derived directly from the invariance of the profile equation (1.4) under the scaling symmetries

$$(Q(\xi),\kappa,\omega) \to (e^{i\theta}Q(\xi),\kappa,\omega)\,, \tag{4.7}$$

$$(Q(\xi),\kappa,\omega) \to (\lambda^{1/\sigma+i\omega/\kappa}Q(\lambda\xi),\lambda^2\kappa,\lambda^2\omega)\,. \tag{4.8}$$

The invariance under the S^1-symmetry (4.7) implies existence of an eigenfunction $Y_1 = iQ$ with the eigenvalue zero, while the symmetry under (4.8) leads (formally) to an eigenfunction $Y_2 = \xi Q' + (1/\sigma + i\omega_0/\kappa_0)Q$ with the eigenvalue $2\kappa_0$. We note that $Y_2(\xi) = \mathrm{O}(\xi^{-1/\sigma-2})$ as $\xi \to \infty$, due to (2.18) and (2.19). We have

$$\mathcal{L}Y_1 = 0\,, \tag{4.9}$$

$$\mathcal{L}Y_2 = 2\kappa_0 Y_2\,. \tag{4.10}$$

We rewrite (4.5) as

$$\frac{\partial w}{\partial \tau} = \mathcal{L}w + \varkappa(\tau)\left(Y_2 - \frac{\omega_0}{\kappa_0}Y_1\right) \tag{4.11}$$

$$\langle \mathcal{J}'(Q), w\rangle = 0\,, \tag{4.12}$$

where the unknowns are $w(\xi,\tau)$ and $\varkappa(\tau)$.

Proceeding further with our formal reasoning, we view (4.11) as a dynamical system with one linear constraint in a suitable linear space $\mathcal{W}$ of functions on $\mathbb{R}^d$. In this formal analysis we will not try to specify $\mathcal{W}$. For a rigorous analysis it would be natural to try to find a suitable Banach space containing all smooth compactly supported functions, together with the functions Y_1 and Y_2. Suppose that the space $\mathcal{W}$ can be decomposed into a direct sum $\mathcal{W} = \mathbb{R}Y_1 \oplus \mathbb{R}Y_2 \oplus \mathcal{Z}$ where $\mathcal{Z} \subset \mathcal{W}$ is invariant under $\mathcal{L}$. In the finite-dimensional situation, a sufficient condition for this would be that the eigenvalues 0 and $2\kappa_0$ are simple. Assuming that such a decomposition exists, we write

$$w(\xi,\tau) = w_1(\tau)Y_1(\xi) + w_2(\tau)Y_2(\xi) + Z(\xi,\tau)\,, \tag{4.13}$$

where w_1, w_2 are scalar functions and $Z(.,\tau) \in \mathcal{Z}$.

After substituting (4.13) into (4.11) and using $\langle \mathcal{J}'(Q), Y_1\rangle = 0$ (a consequence of the invariance of $\mathcal{J}$), we obtain

$$\frac{\partial Z}{\partial \tau} = \mathcal{L}Z \tag{4.14}$$

$$\frac{\partial w_1}{\partial \tau} = -\varkappa(\tau)\frac{\omega_0}{\kappa_0} \tag{4.15}$$

$$\frac{\partial w_2}{\partial \tau} = 2\kappa_0 w_2 + \varkappa(\tau) \tag{4.16}$$

$$w_2 = -\frac{\langle \mathcal{J}'(Q), Z\rangle}{\langle \mathcal{J}'(Q), Y_2\rangle} \tag{4.17}$$

where we assume that $\langle \mathcal{J}'(Q), Y_2\rangle \neq 0$. This is a completely natural assumption in the context of (4.3). From (4.14) - (4.17) one can see that in a finite-dimensional situation and under the assumptions stated above, the following conditions would be equivalent:

(i) If $(w,\varkappa)$ is a solution of (4.5), then $w(\xi,\tau)$ approaches exponentially $aY_1(\xi)$ for some $a \in \mathbb{R}$ and $\varkappa(\tau)$ approaches exponentially zero.

(ii) The spectrum of $\mathcal{L}|_{\mathcal{Z}}$ belongs to the set $\{z \in \mathbb{C} \,|\, \operatorname{Re} z < 0\}$.

In a finite-dimensional situation and under the assumption that the eigenvalues 0 and $2\kappa_0$ are simple, condition (ii) would be equivalent to the condition that all the spectrum of $\mathcal{L}$ except 0 and $2\kappa_0$ lies in $\{z \in \mathbb{C} \,|\, \operatorname{Re} z < 0\}$.

The condition (i) is exactly the linearized orbital stability of the orbit $\left(e^{i\omega_0\tau}Q, \kappa_0\right)$ for the system (4.3), (4.5).

It seems to be a non-trivial problem to put the above formal analysis on a rigorous basis in the infinite-dimensional setting. However, the formal analysis strongly suggests that in a finite dimensional situation which arises in numerical approximations of (4.3) the stability of the solutions $\left(e^{i\omega_0\tau}Q(\xi), \kappa_0\right)$ should be governed by the spectrum of an appropriate approximation of the operator $\mathcal{L}$.

We remark that our stability analysis is independent of $\mathcal{J}$, except for the natural assumption $\langle \mathcal{J}'(Q), Y_2\rangle \neq 0$.

We numerically calculated the approximation of eigenvalues for a discrete approximation of $\mathcal{L}$ *in the space of radial functions* along the branches of solutions parameterized by ϵ. The original system (4.3) was truncated to a finite interval $(0, \xi_1)$ by imposing the time-dependent boundary condition at $\xi = \xi_1$

$$\frac{\partial v}{\partial \tau}(\xi_1, \tau) + \kappa(\tau)\xi_1 \frac{\partial v}{\partial \xi}(\xi_1, \tau) + \frac{\kappa(\tau)}{\sigma} v(\xi_1, \tau) = 0\,, \tag{4.18}$$

which is a time-dependent equivalent of the boundary condition derived from the asymptotic expansion in Section 2. The condition (4.18) was also used in the numerical simulations in [4] and [5]. The computations were carried out for $\xi_1 = 30$.

The accuracy of our numerical approximation can be checked indirectly by comparing the predicted eigenvalues 0 and $2\kappa_0$ with the corresponding eigenvalues we obtained from our numerical calculations. We saw a very good agreement, in most cases the error was of the order 10^{-4}. Both 0 and $2\kappa_0$ appeared simple, except in some natural degenerate cases when other eigenvalues were crossing them as we moved along branches.

Case I ($d = 1$, $\sigma = 2.3$, $\delta = 0$): The calculations confirmed what one intuitively expects: The solutions on the upper part of the branch $j = 1$ are stable and all other solutions are unstable. In Figure 3.1 we used a solid line for the stable parts of the curves and a dashed line for the unstable parts.

Case II ($d = 3$, $\sigma = 1$, $\delta = 0$): The situation is similar with one notable exception. In our computations we detected stable solutions also on the lower part of the branch $j = 2$. Accuracy of our approximation did not allow us to decide whether all solutions on the lower part of the branch $j = 2$ for $0 < \epsilon < \epsilon^*$ are stable, since as ϵ approached zero we observed some eigenvalues very close to the imaginary axis. In Figure 3.6 we used a solid line for the stable parts of the curves and a dashed line for the unstable parts.

We emphasize again that the calculations we carried out only deal with stability in the space of radial functions. Based on computations in [5], it appears that the solutions on the upper part of the branch $j = 1$ are also stable with respect to perturbations that break the radial symmetry.

It is clear that further work is required to fully clarify issues concerning the stability of solutions described in this paper.

5. Appendix. In this section we give a full proof of Theorem 2.1.

We fix $r_1 > 0$ and $\vartheta > 0$. Values of these parameters will be chosen later. We denote by $\mathcal{X}_\vartheta$ the Banach space of continuous functions $u : [\xi_1, \infty) \to \mathbb{C}$ for which the norm

$$\|u\|_\vartheta = \sup_{\xi \geq \xi_1} |\xi|^{1/\sigma - \vartheta} |u(\xi)|$$

is finite. In this proof we use the following notation

$$\tilde{\Lambda} = [\kappa_1, \kappa_2] \times [\omega_1, \omega_2] \times [0, \epsilon_1] \times [\delta_1, \delta_2]$$
$$\Lambda = \{\gamma \in \mathbb{C} \,|\, |\gamma| \leq r_1\} \times \tilde{\Lambda}\,.$$

A point in $\tilde{\Lambda}$ is denoted by $\tilde{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \equiv (\kappa, \omega, \epsilon, \delta)$ and similarly $\lambda \in \Lambda$ represents $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \equiv (\gamma, \kappa, \omega, \epsilon, \delta)$. The functions P, E, W, and K introduced

in Section 2 are written as $P = P(\tilde{\lambda}, \xi)$, $E = E(\tilde{\lambda}, \xi)$, $W = W(\tilde{\lambda}, \xi, \eta)$, $K = K(\tilde{\lambda}, \xi)$. With a slight abuse of notation we sometimes write also $P = P(\lambda, \xi)$ even if the function does not depend on λ_0.

LEMMA 5.1. *There exists $C > 0$ such that for $\tilde{\lambda} \in \tilde{\Lambda}$, $\xi, \eta \geq \xi_0$ we have*

$$|K(\tilde{\lambda}, \xi, \eta)| \leq \begin{cases} C\xi^{-1/\sigma}\eta^{1/\sigma - 1} & \text{for } \xi_0 \leq \eta \leq \xi\,, \\ C\xi^{-d+1/\sigma}\eta^{-1-1/\sigma+d} & \text{for } \xi_0 \leq \xi \leq \eta\,. \end{cases} \tag{5.1}$$

PROOF: This statement is an easy consequence of the definition of K and of the asymptotic expansion for U as $z \to \infty$ in the sector $-\pi/2 \leq \arg z \leq \pi$. An important point in connection with (2.6) is that the constant in the remainder $\mathrm{O}(|z|^{-n})$ can be taken same when the parameter a runs through a compact subset of $\{z \,|\, \mathrm{Re}\, z > 0\}$. (This can be seen for example from (2.5).)□

LEMMA 5.2. *Assume $(2\sigma + 1)\vartheta < 2 + 2/\sigma - d$. Then the formula*

$$T(\lambda, u)(\xi) = \gamma P(\tilde{\lambda}, \xi) - \int_{\xi_1}^{\infty} (1 + i\delta) K(\tilde{\lambda}, \xi, \eta) |u(\eta)|^{2\sigma} u(\eta)\, d\eta \tag{5.2}$$

defines a continuous mapping $T : \Lambda \times \mathcal{X}_\vartheta \to \mathcal{X}_\vartheta$.

Moreover, there exists $C > 0$ such that

$$\|T(\lambda, u)\|_\vartheta \leq C|\gamma|\xi_1^{-\vartheta} + C\xi_1^{-2+2\sigma\vartheta} \|u\|_\vartheta^{2\sigma+1} \tag{5.3}$$

and

$$\|T(\lambda, u) - T(\lambda, v)\|_\vartheta \leq C\xi_1^{-2+2\sigma\vartheta} \|u - v\|_\vartheta \left(\|u\|_\vartheta^{2\sigma} + \|v\|_\vartheta^{2\sigma} \right) \tag{5.4}$$

for all $\lambda \in \Lambda$ and all $u, v \in \mathcal{X}_\vartheta$.

PROOF: The convergence of the integrals in (5.2) and (5.3) follows easily from Lemma 5.1. To get the estimate (5.4) we use Lemma 5.1 together with the elementary inequality

$$\left| |z_1|^{2\sigma} z_1 - |z_2|^{2\sigma} z_2 \right| \leq M |z_1 - z_2| \left(|z_1|^{2\sigma} + |z_2|^{2\sigma} \right)\,,$$

which holds for all $z_1, z_2 \in \mathbb{C}$ if $M > 0$ is properly chosen.

The continuity of the mapping T can be proved by suitably splitting the integral over (ξ_1, ∞) into two integrals, one over (ξ_1, ξ_2) and the other one over (ξ_2, ∞). We can then estimate the integral over (ξ_2, ∞) by using Lemma 5.1. □

The estimates (5.3) and (5.4) show that $T(\lambda, .)$ is a contraction of the ball $\mathcal{B}_\rho = \{u \in \mathcal{X}_\vartheta \,|\, \|u\|_\vartheta \leq \rho\}$ into itself if ρ is such that

$$Cr_1\xi_1^{-\vartheta} + C\xi_1^{-2+2\sigma\vartheta}\rho^{2\sigma+1} \leq \rho\,, \tag{5.5}$$

$$2C\rho^{2\sigma}\xi_1^{-2+2\sigma\vartheta} < 1\,. \tag{5.6}$$

PROPOSITION 5.3. *Assume $(2\sigma+1)\vartheta < 2+2/\sigma-d$. If (5.5) and (5.6) are satisfied, then for each $\lambda \in \Lambda$ the mapping $T(\lambda, .)$ has a unique fixed point $u(\lambda) = u(\lambda, \xi)$ in $\mathcal{B}_\rho = \{u \in \mathcal{X}_\vartheta \,|\, \|u\|_\vartheta \leq \rho\}$.*

Moreover, the mapping $\lambda \mapsto u(\lambda)$ is a continuous mapping from Λ to $\mathcal{X}_\vartheta$.

PROOF: The proposition follows directly from Lemma 5.2 and the Banach Fixed Point Theorem. □

Next we need to establish that the solutions $u(\lambda, \xi)$ have the required regularity properties at the infinity.

LEMMA 5.4. *The function $u(\lambda) = u(\lambda, \xi)$ from Proposition 5.3 exhibits the following behavior as $\xi \to \infty$:*

$$u(\lambda,\xi) = \xi^{-1/\sigma - i\omega/\kappa}\left(\sum_{j=0}^{n} a_j(\lambda)\xi^{-2j} + \mathrm{O}(\xi^{-2n-2})\right), \tag{5.7}$$

where $a_j(\lambda)$ are continuous functions of λ and the constant in $\mathrm{O}(\xi^{-2n-2})$ is independent of $\lambda \in \Lambda$.

Moreover, the expansion (5.7) can be differentiated in the following sense:

$$\frac{\partial^k u}{\partial \xi^k}(\lambda,\xi) = \frac{\partial^k}{\partial \xi^k}\left(\xi^{-1/\sigma - i\omega/\kappa}\sum_{j=0}^{n} a_j(\lambda)\xi^{-2j}\right) + \mathrm{O}(\xi^{-1/\sigma-2n-2-k}), \tag{5.8}$$

where the constant in $\mathrm{O}(\xi^{-1/\sigma-2n-2-k})$ is independent of $\lambda \in \Lambda$.

PROOF: The proof is similar to bootsrapping arguments used in the regularity theory. From (5.2) and the assumption $(2\sigma+1)\vartheta < 2 + 2/\sigma - d$ we get immediately

$$u(\lambda,\xi) = \mathrm{O}(\xi^{-1/\sigma}) \quad \text{as} \quad \xi \to \infty. \tag{5.9}$$

We now rewrite the equation $u = Tu$ as

$$u(\xi) = (\gamma+\gamma_1)P(\xi) - \frac{1+i\delta}{1-i\epsilon}P(\xi)\int_\xi^\infty EW^{-1}|u|^{2\sigma}u + \frac{1+i\delta}{1-i\epsilon}E(\xi)\int_\xi^\infty PW^{-1}|u|^{2\sigma}u\,, \tag{5.10}$$

where

$$\gamma_1 = \frac{1+i\delta}{1-i\epsilon}\int_{\xi_1}^\infty EW^{-1}|u|^{2\sigma}u\,.$$

Using (5.9) and (5.10), we obtain (for a suitable a_0)

$$u(\xi) = \xi^{-1/\sigma-i\omega/\kappa}\left(a_0 + \mathrm{O}(\xi^{-2})\right) \tag{5.11}$$

We now repeat the procedure and use (5.10) with (5.11) instead of (5.9). After integrating by parts in integrals of the form

$$\int_\xi^\infty \eta^\alpha e^{-\nu\eta^2/2} = -\frac{1}{\nu}\int_\xi^\infty \eta^{\alpha-1}\frac{\partial}{\partial\eta}e^{-\nu\eta^2/2}$$

which come up in the calculation, we get easily

$$u(\xi) = \xi^{-1/\sigma-i\omega/\kappa}\left(a_0 + a_1\xi^{-2} + \mathrm{O}(\xi^{-4})\right).$$

Repeating this procedure, we get (5.7) by induction. The expansion (5.8) can be obtained in a similar way, after we differentiate (5.10). □

LEMMA 5.5. *The mapping $\lambda \mapsto u(\lambda)$ from Proposition 5.3 is continuously differentiable as a mapping from Λ to $\mathcal{X}_\vartheta$*

PROOF: We define $\phi : \Lambda \times \mathcal{X}_\vartheta \to \mathcal{X}_\vartheta$ as $\phi(\lambda, v) = v - T(\lambda, v)$ and we denote $\mathcal{M} = \{(\lambda, u(\lambda)), \lambda \in \Lambda\} \subset \Lambda \times \mathcal{X}_\vartheta$. By using similar estimates as those used in the proof of Lemma 5.2, one can easily see that ϕ is differentiable with respect to v and

that the (partial) derivative $\mathrm{D}_v\phi$ is continuous as a map from $\Lambda \times \mathcal{X}_\vartheta$ into the space $\mathcal{L}(\mathcal{X}_\vartheta, \mathcal{X}_\vartheta)$ of bounded linear operators on $\mathcal{X}_\vartheta$. Moreover, a calculation similar to the one leading to (5.4) gives

$$\|I - \mathrm{D}_v\phi\| \leq 2C\xi_1^{-2+2\sigma\vartheta}\rho^{2\sigma} \tag{5.12}$$

in $\Lambda \times \mathcal{B}_\rho$. Therefore, since we assume (5.6), $\mathrm{D}_v\phi$ is invertible for all $(\lambda, v) \in \Lambda \times \mathcal{B}_\rho$. By Lemma 5.6 below, ϕ is also differentiable with respect to λ at each point of $\mathcal{M}$ and the partial derivative $\mathrm{D}_\lambda\phi(\lambda, v)$ is continuous in $\mathcal{M}$. Now we can conclude with the same arguments as in the Implicit Function Theorem to show that u is differentiable and

$$\frac{\partial u}{\partial \lambda}(\lambda) = -\left[\mathrm{D}_v\phi(\lambda, u(\lambda))\right]^{-1} \cdot \left[\mathrm{D}_\lambda\phi(\lambda, u(\lambda))\right] . \tag{5.13}$$

□

Lemma 5.5 also implies the differentiability of the function $\lambda \mapsto \frac{\partial u}{\partial \xi}(\lambda, \xi_1)$, since the partial derivative of u at ξ_1 can be expressed in terms of u due to the fact that u solves a differential equation. For example, one can take a smooth function $\varphi : [\xi_1, \infty) \to \mathbb{R}$ with $\varphi(\xi_1) = 1$ and $\varphi(\xi) \equiv 0$, for $\xi > \xi_1 + 1$ and write $u(\xi_1) = -\int_{\xi_1}^{\infty} (u\varphi)''\, d\xi$. Then we express u'' from the differential equation for the profile (1.4) and integrate by parts to obtain

$$\frac{\partial u}{\partial \xi}(\xi_1) = c_1 u(\xi_1) + \int_{\xi_1}^{\infty} \left(u(\xi)\psi(\xi) + |u(\xi)|^{2\sigma} u(\xi)\tilde{\psi}(\xi)\right) d\xi\,,$$

where ψ, $\tilde{\psi}$ are supported on a finite interval and c_1 is a suitable constant. □

Lemma 5.6. *The mapping $\phi : \Lambda \times \mathcal{X}_\vartheta \to \mathcal{X}_\vartheta$ defined as $\phi(\lambda, v) = v - T(\lambda, v)$ is differentiable with respect to λ at each point $\mathcal{M} = \{(\lambda, u(\lambda)), \lambda \in \Lambda\} \subset \Lambda \times \mathcal{X}_\vartheta$. Moreover, the partial derivative $\mathrm{D}_\lambda\phi$ is continuous in $\mathcal{M}$.*

Proof: Using (2.3) we note that

$$\frac{\partial U}{\partial a}(a, b, z) = z^{-a} \log z \left(1 + \mathrm{O}(|z|^{-1})\right) , \tag{5.14}$$

as $z \to \infty$ in the sector $-\pi/2 \leq \arg z < \pi$ and a belongs to a compact subset of $\{\zeta \,|\, \mathrm{Re}\,\zeta > 0\} \subset \mathbb{C}$. Let us carry out the proof for the partial derivative $\partial\phi/\partial\lambda_1$, for example. The other partial derivatives can be handled in a similar way. (If fact, for λ_0,λ_2 and λ_4 the proof is much easier.) When $\epsilon > 0$ one sees easily (using the rapid decay of $W^{-1}(\eta)$ in that case) that ϕ is differentiable in λ_1 and that

$$\frac{\partial \phi}{\partial \lambda_1}(\lambda, v)(\xi) = \gamma\frac{\partial P}{\partial \lambda_1}(\lambda, \xi) + \int_{\xi_1}^{\infty} \frac{\partial K}{\partial \lambda_1}(\lambda, \xi, \eta)\, |v(\eta)|^{2\sigma} v(\eta)\, d\eta\,. \tag{5.15}$$

It follows from (5.14) that, for any $\vartheta > 0$, the first term on the right-hand side is continuous as a mapping from Λ to $\mathcal{X}_\vartheta$. For any $\epsilon_2 > 0$, the integral on the right-hand side has the required continuity properties in the region $\epsilon_2 \leq \epsilon \leq \epsilon_1$. due to the rapid decay of $W^{-1}(\eta)$. The problem is to obtain estimates which are uniform for $0 < \epsilon < \epsilon_1$. A closer inspection of the integral on the right-hand side of (5.15) reveals that the only terms for which the required continuity is not obvious come from differentiating the exponential $e^{-i\kappa\eta^2/2(1-i\epsilon)}$ that appears in $W^{-1}(\eta)$ This term can

be handled with using integration by parts as follows. Recalling that $\lambda_1 = \kappa$, we must estimate an integral of the form

$$\int_{\xi}^{\infty} P(\lambda,\eta)\eta^{d-1}\frac{\partial}{\partial\kappa}\left(e^{\frac{i\kappa}{1-i\epsilon}\frac{\eta^2}{2}}\right)|v(\eta)|^{2\sigma}v(\eta)\,d\eta\,. \tag{5.16}$$

To estimate this integral we write

$$\frac{\partial}{\partial\kappa}\left(e^{\frac{i\kappa}{1-i\epsilon}\frac{\eta^2}{2}}\right) = \frac{\eta}{2\kappa}\frac{\partial}{\partial\eta}\left(e^{\frac{i\kappa}{1-i\epsilon}\frac{\eta^2}{2}}\right)$$

and integrate by parts. This eliminates the power η^2 obtained after differentiating in (5.16), and the required estimate follows. □

To complete the proof of Theorem 2.1 we need to show that the coordinate $\gamma = \lambda_0$ can be replaced by $\beta = u(\lambda,\xi_1)$, if r_1 and ρ are chosen properly. We have

$$\frac{\partial u}{\partial\lambda_0}(\lambda,\xi_1) = P(\lambda,\xi_1) + \int_{\xi_1}^{\infty} K(\tilde\lambda,\xi_1,\eta)\frac{\partial}{\partial\lambda_0}|u|^{2\sigma}u(\lambda,\eta)\,d\eta\,. \tag{5.17}$$

(Since λ_0 is complex, we interpret this, with a slight abuse of notation, as an equation between real 2×2 matrices. Another possibility would be to interpret it literally and do a similar calculation for $\partial/\partial\bar\lambda_0$.) We carry out the differentiation in the integral on the right-hand side and use (5.13), which gives

$$\frac{\partial u}{\partial\lambda_0}(\lambda) = -\left[\mathrm{D}_v\phi(\lambda,u(\lambda))\right]^{-1}\cdot P\,. \tag{5.18}$$

We note from (5.12) that by a suitable choice of the constant ρ (to be specified below) we can achieve that

$$\|[\mathrm{D}_v\phi(\lambda,u(\lambda))]^{-1}\| \le 2\,, \tag{5.19}$$

in Λ where the norm is taken in the space of linear operators on the space $\mathcal{X}_0$. Using (5.17) - (5.19), we see easily, by a similar calculation as in the proof of (5.4), that by a suitable choice of ρ we can achieve that $\frac{\partial u}{\partial\lambda_0}(\lambda,\xi_1)/P(\lambda,\xi_1)$ is close to the identity uniformly in Λ. A suitable choice for the constants ρ and r_1 is, for example, as follows

$$\rho = \alpha\xi_1^{1/\sigma-\vartheta}\,, \qquad r_1 = \alpha^{2\sigma+1}\xi_1^{1/\sigma}\,,$$

where α is sufficiently small. The proof of Theorem 2.1 can now be easily completed. □

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